I reordered!

Solution to ST 793 Midterm Exam, 2018

4a. $r = f(\hat{\sigma}_X^2, \hat{\sigma}_Y^2, \hat{\sigma}_{12}) = \dfrac{\hat{\sigma}_{12}}{\sqrt{\hat{\sigma}_X^2 \hat{\sigma}_Y^2}}$ which is a continuous fn. for $\hat{\sigma}_X^2 > 0$, $\hat{\sigma}_Y^2 > 0$

$$\hat{\sigma}_{12} = \underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_X)(Y_i - \mu_Y)}_{\downarrow P \text{ by WLLN} \atop \sigma_{12}} - \underbrace{(\bar{X} - \mu_X)(\bar{Y} - \mu_Y)}_{\downarrow P \atop 0}$$

by $o_p(1)\, o_p(1) = o_p(1)$ and WLLN for $(\bar{X} - \mu_X)$ + $(\bar{Y} - \mu_Y)$

$$\hat{\sigma}_X^2 = \underbrace{\frac{1}{n}\sum (X_i - \mu_X)^2}_{\downarrow P \text{ by WLLN} \atop \sigma_X^2} - \underbrace{(\bar{X} - \mu_X)^2}_{\downarrow P \atop 0}$$

$\downarrow P$, since $\bar{X} - \mu \xrightarrow{P} 0$ by WLLN + $(\ )^2$ is a cont. fn.

$\xrightarrow{P} \sigma_X^2$ since $x + y$ is a cont. fn.

So by the cont. theorem, $r \xrightarrow{P} \rho = \dfrac{\sigma_{12}}{\sigma_X \sigma_Y}$.

4b. $s^2 \xrightarrow{P} Var(X_1 - Y_1) = \sigma_1^2$ by same arg. as above along with $\left(\frac{n-1}{n}\right) \to 1$

$\sqrt{n}(\bar{X} - \bar{Y}) = \sqrt{n}\, \frac{1}{n}\sum (X_i - Y_i)$

$\xrightarrow[\text{under } H_0]{d} N(0, \sigma_d^2)$ by CLT applied to $D_i = X_i - Y_i$

thus $\dfrac{\sqrt{n}(\bar{X} - \bar{Y})}{s} = \dfrac{\sqrt{n}(\bar{X} - \bar{Y})}{\sigma}\left(\dfrac{\sigma}{s}\right) \xrightarrow{d} N(0,1)$ by Slutsky since $\dfrac{\sigma}{s} \xrightarrow{P} 1$ by cont. theorem

5a. $L(p) = \dfrac{n!}{n_1!\, n_2!\, n_3!}\, p^{N_1} p^{N_2} (1 - 2p)^{n - N_1 - N_2}$ $\quad \log L = C + (N_1 + N_2)\log p + (n - N_1 - N_2)\log(1 - 2p)$

$$\frac{d}{dp}(\log L) = \frac{N_1 + N_2}{p} - \frac{2(n - N_1 - N_2)}{1 - 2p} = 0$$

$$N_1 + N_2 - \cancel{2p(N_1 + N_2)} - 2np + \cancel{2p(N_1 + N_2)} = 0$$

$\hat{p} = \dfrac{N_1 + N_2}{2n}$ so $\hat{p}_1 = \hat{p}_2 = \hat{p}$

$\hat{p}_3 = 1 - \hat{p}_1 - \hat{p}_2 = 1 - 2\hat{p}$

5b. To get score $\underset{\sim}{S}$, we need $\log L(p_1,p_2|\underset{\sim}{N}) = C + N_1 \log p_1 + N_2 \log p_2 + (n-N_1-N_2)\log(1-p_1-p_2)$

$$S_1(p_1,p_2) = \frac{\partial}{\partial p_1}(\log L) = \frac{N_1}{p_1} - \frac{(n-N_1-N_2)}{1-p_1-p_2} \text{ or } \frac{N_1}{p_1} - \frac{N_3}{p_3}$$

$$S_2(p_1,p_2) = \frac{\partial}{\partial p_2} = \frac{N_2}{p_2} - \frac{N_3}{p_3}$$

Thus

$$T_S = \left(\frac{N_1}{\tilde{p}} - \frac{N_3}{1-2\tilde{p}}, \frac{N_2}{\tilde{p}} - \frac{N_3}{1-2\tilde{p}}\right) \frac{1}{n} \begin{pmatrix} \tilde{p}(1-\tilde{p}) & -\tilde{p}^2 \\ -\tilde{p}^2 & \tilde{p}(1-\tilde{p}) \end{pmatrix} \begin{pmatrix} \frac{N_1}{\tilde{p}} - \frac{N_3}{1-2\tilde{p}} \\ \frac{N_2}{\tilde{p}} - \frac{N_3}{1-2\tilde{p}} \end{pmatrix}$$

5c. $h(p_1,p_2) = p_1 - p_2 \quad H(p_1,p_2) = (1,-1)$

$$T_W = (\hat{p}_1 - \hat{p}_2) \left\{ (1,-1) \frac{1}{n} \begin{pmatrix} \hat{p}_1(1-\hat{p}_1) & -\hat{p}_1\hat{p}_2 \\ -\hat{p}_1\hat{p}_2 & \hat{p}_2(1-\hat{p}_2) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1} (\hat{p}_1 - \hat{p}_2)$$

$$\left\{ \frac{1}{n} \left(\hat{p}_1(1-\hat{p}_1) + \hat{p}_1\hat{p}_2, -\hat{p}_1\hat{p}_2 - \hat{p}_2(1-\hat{p}_2) \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}^{-1}$$

$$= \frac{n(\hat{p}_1 - \hat{p}_2)^2}{\hat{p}_1(1-\hat{p}_1) + \hat{p}_2(1-\hat{p}_2) + 2\hat{p}_1\hat{p}_2} \qquad \hat{p}_1 = \frac{N_1}{n} \quad \hat{p}_2 = \frac{N_2}{n}$$

1. $Y_n = O_p(1)$ means "For each $\epsilon > 0$, there exists M_ϵ + n_ϵ such that $P(|Y_n| > M_\epsilon) < \epsilon$ for all $n \ge n_\epsilon$.

2. $\prod_{i=1}^{n} f_e\left(\frac{Y_i - x_i^T \beta}{\sigma g(x_i,\tau)}\right) \frac{1}{\sigma g(x_i,\tau)}$

3. only b is true
 a. never put an n on the rhs of a limit
 c. AN says nothing about moments