Solutions, ST 793 Final Exam, 2017

1\. a) $\displaystyle\int \frac{\partial}{\partial\theta} \log f(y;\theta)\Big|_{\theta=\theta_0} g(y)\,dy = 0$

b) i) est. $I(\theta)$, parametric bootstrap
ii) sandwich, jackknife, nonpar. bootstrap

2\. $\bar{q}_1 = \frac{1}{n}\sum_{i=1}^n q_1(Y_i) \xrightarrow{P} 0$ by WLLN since $E\{q_1(Y_1)\}=0$

$\bar{q}_2 = \frac{1}{n}\sum_{i=1}^n q_2(Y_i) \xrightarrow{P} \mu_2$ by WLLN since $E\{q_2(Y_1)\}=\mu_2$

$\begin{pmatrix}\bar{q}_1\\ \bar{q}_2\end{pmatrix} \xrightarrow{P} \begin{pmatrix}0\\ \mu_2\end{pmatrix}$ jointly because joint convergence in prob. + marg. convergence of components are equivalent

$\hat{\theta}_1 \xrightarrow{P} 0\cdot\mu_2 = 0$ by continuity theorem since $x_1 x_2$ is continuous

$\hat{\theta}_2 \xrightarrow{P} \mu_{12}$ by WLLN since $E\, q_1(Y_1) q_2(Y_1) = \mu_{12}$

$\hat{\theta}_3 \xrightarrow{P} \frac{0}{\mu_2} = 0$ by cont. theorem since x_1/x_2 is cont. if $x_2 \neq 0$.

b) For $\hat{\theta}_1$, $\psi_1(Y_i,\underset{\sim}{\theta}) = q_1(Y_i)\theta_5 - \theta_1$, $\psi_2(Y_i,\theta_5) = q_2(Y_i) - \theta_5$, $\theta = \begin{pmatrix}\theta_1\\ \theta_5\end{pmatrix}$

or $\psi_1(Y_i,\underset{\sim}{\theta}) = \theta_4\theta_5 - \theta_1$
$\psi_2(Y_i,\underset{\sim}{\theta}) = q_1(Y_i) - \theta_4$ $\quad \theta = \begin{pmatrix}\theta_1\\ \theta_4\\ \theta_5\end{pmatrix}$
$\psi_3(Y_i,\underset{\sim}{\theta}) = q_2(Y_i) - \theta_5$

For $\hat{\theta}_2$, $\psi(Y_i,\theta_2) = q_1(Y_i)q_2(Y_i) - \theta_2$

For $\hat{\theta}_3$, $\psi_1(Y_i,\underset{\sim}{\theta}) = q_1(Y_i)/\theta_5 - \theta_1$, $\psi_2(Y_i,\theta_5) = q_2(Y_i) - \theta_5$ $\quad \theta = \begin{pmatrix}\theta_1\\ \theta_5\end{pmatrix}$

or same as for $\hat{\theta}_1$ with $\psi_1(Y_i,\underset{\sim}{\theta}) = \theta_4/\theta_5 - \theta_3$ $\quad \theta = \begin{pmatrix}\theta_3\\ \theta_4\\ \theta_5\end{pmatrix}$

3. a) $m_3 = \frac{1}{n}\Sigma(X_i - \bar{X})^3$

$\hat{\theta}_1 = \bar{X}$ $\quad g(X_i, \mu) = (X_i - \mu)^3$ $\quad E\,g(X_i, \theta) = \mu_3$

$\theta = \mu$ $\quad g'(X_i, \mu) = -3(X_i - \mu)^2$ $\quad E[g'(X_i, \theta)] = -3\sigma^2$

By Thm 5.28, $h_T(X_i) = (X_i - \mu)^3 - \mu_3 + (-3\sigma^2)(X_i - \mu)$

$= (X_i - \mu)^3 - \mu_3 - 3\sigma^2(X_i - \mu)$

b) $h_1(X_i) = (X_i - \mu)^2 - \sigma^2$ for $S_n^2 = \hat{\theta}_1$ $\quad \theta_1 = \sigma^2$

$h_2(X_i)$ is from m_3 $\quad m_3 = \hat{\theta}_2$ $\quad \theta_2 = \mu_3$

$\text{Skew} = \hat{\theta}_2 / \hat{\theta}_1^{3/2} = g(\hat{\underline{\theta}})$ $\quad \frac{\partial g}{\partial \theta_1} = \frac{(-3/2)\theta_2}{\theta_1^{5/2}}$ $\quad \frac{\partial g}{\partial \theta_2} = \frac{1}{\theta_1^{3/2}}$

By Theorem 5.27

$$h_T(X_i) = \frac{-3\theta_2}{2\theta_1^{5/2}}\left[(X_i - \mu)^2 - \sigma^2\right] + \frac{1}{\theta_1^{3/2}} h_2(X_i)$$

$$= \frac{-3\mu_3}{2\sigma^5}\left[(X_i - \mu)^2 - \sigma^2\right] + \frac{1}{\sigma^3} h_2(X_i)$$

4.

standard	new	a)
1,2	3,4,5	W = 12
1,3	2,4,5	11
1,4	2,3,5	10
1,5	2,3,4	9
3,2	1,4,5	10
4,2	1,3,5	9
5,2	1,3,4	8
3,4	1,2,5	8
3,5	1,2,4	7
4,5	1,2,3	6

b)

W	Prob
6	1/10
7	1/10
8	2/10
9	2/10
10	2/10
11	1/10
12	1/10

c) $P(W \geq 12) = \frac{1}{10}$

5. a) $\sqrt{n}(\bar{Y}-\bar{X}) = \sqrt{n}(\bar{Y}-\lambda_1-(\bar{X}-\lambda_2))$ assuming $\lambda_1=\lambda_2$

$\xrightarrow{d} \sigma_1 Z_1 + \sigma_2 Z_2$ where Z_1 & Z_2 are indep. standard normal

by CLT & independence

$\bar{X} \xrightarrow{P} \lambda_1$, $\bar{Y} \xrightarrow{P} \lambda_2$ by WLLN

$$T_W = \frac{n(\bar{Y}-\bar{X})^2}{\bar{Y}+\bar{X}} \xrightarrow{d} \frac{(\sigma_1 Z_1+\sigma_2 Z_2)^2}{\lambda_1+\lambda_2}$$ by cont. thm & Slutsky

$\mathrm{Var}(\sigma_1 Z_1+\sigma_2 Z_2) = \sigma_1^2+\sigma_2^2$ so $\sigma_1 Z_1+\sigma_2 Z_2 \overset{d}{=} \sqrt{\sigma_1^2+\sigma_2^2}\, Z$

So $$T_W \xrightarrow{d} \frac{(\sigma_1^2+\sigma_2^2)Z^2}{2\lambda}$$

$\lambda_1=\lambda_2=\lambda$

b) $$T_{GW} = \frac{(\bar{Y}-\bar{X})^2}{\left(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}\right)}$$ this is the Welch t^2 $\xrightarrow{d} \chi_1^2$ under H_0

6. $$se(n\,MSE(\hat{\theta}_1)) = n\sqrt{\frac{1}{N-1}\sum_{j=1}^{N}\left[(\hat{\theta}_{1,j}-\theta)^2 - \widehat{MSE}(\hat{\theta}_1)\right]^2 / N}$$

and similar for $n\,MSE(\hat{\theta}_2)$. $MSE(\hat{\theta})$ is a sample mean!

4th column is just a ratio of means, from 7.2.3

$$se(\underbrace{\widehat{MSE}(\hat{\theta}_1)/\widehat{MSE}(\hat{\theta}_2)}_{\widehat{\text{ratio}}}) = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N}\left[(\hat{\theta}_{1,j}-\theta)^2 - \widehat{\text{ratio}}\,(\theta_{2,j}-\theta)^2\right]^2}$$