Solutions - Final Exam - ST 793 2018

1a. $\int \left[\frac{\partial}{\partial\theta} \log f(y;\underset{\sim}{\theta}_0)\right] g(y)\, dy = 0$ defining Equation for M-estimation

b(i) $I(\hat{\underset{\sim}{\theta}})^{-1}/n$, $I_T(\underset{\sim}{Y},\hat{\underset{\sim}{\theta}})^{-1} = \bar{I}(\underset{\sim}{Y},\hat{\underset{\sim}{\theta}})^{-1}/n$

b(ii)

$$\hat{V}_J = \frac{1}{n}\frac{1}{n-1}\sum_{i=1}^{n}\left(\hat{\underset{\sim}{\theta}}_{ps,i} - \underset{\sim}{\theta}_J\right)\left(\hat{\theta}_{ps,i} - \underset{\sim}{\theta}_J\right)^T \qquad \hat{\underset{\sim}{\theta}}_{ps,i} = n\hat{\theta} - (n-1)\hat{\theta}_{[i]}$$

$$\hat{\underset{\sim}{\theta}}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{ps,i}$$

$$\hat{V}_B = \frac{1}{n-1}\sum_{i=1}^{n}\left(\hat{\underset{\sim}{\theta}}_i^* - \bar{\underset{\sim}{\theta}}^*\right)\left(\hat{\underset{\sim}{\theta}}_i^* - \bar{\underset{\sim}{\theta}}^*\right)^T$$

where $\hat{\underset{\sim}{\theta}}_i^* = \hat{\theta}(Y_1^*,\dots,Y_n^*)$ where $Y_1^*,\dots,Y_n^*$ are iid from $F_n(y)$ = empirical dist. fn or with replacemt from $\{Y_1,\dots,Y_n\}$

2a) $(\bar{X},\bar{Y})^T$ is $AN\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}/n\right)$ as $n\to\infty$ by the Mult. CLT

b) By a) + the continuity theorem for $\xrightarrow{d}$ since xy is cont. $T = (\sqrt{n}\,\bar{X})(\sqrt{n}\,\bar{Y}) \xrightarrow{d} Q$ where Q is the product of two correlated normal rv's with cov. $\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}$

c) $\sqrt{n}\,T = (\sqrt{n}\,\bar{X})(\bar{Y}) \xrightarrow{d} N(0,\sigma_1^2)\mu_2 = N(0,\mu_2^2\sigma_1^2)$ by a) + Slutsky's Theorem. (or Delta)

3a) By the mult. approximation by averages theorem and Theorem 5.25

$$\begin{pmatrix}\hat{\eta}_{1/4}\\ \hat{\eta}_{3/4}\end{pmatrix} \text{ is } AN\left(\begin{pmatrix}\eta_{1/4}\\ \eta_{3/4}\end{pmatrix}, \; n\Sigma = \begin{pmatrix}\frac{3}{16\,F'(\eta_{1/4})^2} & \frac{1}{16\,F'(\eta_{1/4})F'(\eta_{3/4})}\\ & \frac{3}{16\,F'(\eta_{3/4})^2}\end{pmatrix}\right)$$

b) $h_T(X_i) = \left[\frac{3/4 - I(X_i \le \eta_{3/4})}{F'(\eta_{3/4})}\right] - \left[\frac{1/4 - I(X_i \le \eta_{1/4})}{F'(\eta_{1/4})}\right]$

c) $\frac{1}{n}\left(\frac{3}{16 F'(\eta_{3/4})^2} + \frac{3}{16 F'(\eta_{1/4})^2} - \frac{2}{16 F'(\eta_{3/4}) F'(\eta_{1/4})}\right)$

	Y = new		X = old		Rank X	Rank Y	W_p
4.a)	25	32	33	45	1, 2	3, 4	3 = W
	33	32	25	45	3, 2	1, 4	5
	45	32	33	25	4, 2	3, 1	6
	25	33	32	45	1, 3	2, 4	4
	25	45	33	32	1, 4	3, 2	5
	33	45	25	32	3, 4	1, 2	7

b) Perm. Dist $W = 3 \quad 4 \quad 5 \quad 6 \quad 7$

$P = 1/6 \quad 1/6 \quad 1/3 \quad 1/6 \quad 1/6$

c) $P(W \le 3) = 1/6$

5. $\widehat{Bias}(\hat\theta) = \frac{1}{N}\sum[\hat\theta_i - \theta_0]$ so $SE = \sqrt{\frac{1}{N-1}\sum(\hat\theta_i - \bar{\hat\theta})^2 / N}$

$\widehat{Var}(\hat\theta) = s^2$ of $\hat\theta_i$ so $SE = \sqrt{\frac{\hat\mu_4 - [\widehat{Var}(\hat\theta)]^2}{N}}$ $\quad \hat\mu_4 = \frac{1}{N}\sum(\hat\theta_i - \bar{\hat\theta})^4$

$\hat{E}[\widehat{Var}(\hat\theta)] = \frac{1}{N}\sum \widehat{Var}(\theta_i)$ so $SE = \sqrt{\frac{\frac{1}{N-1}\sum(\widehat{Var}(\theta_i) - \hat{E}[\widehat{Var}(\theta)])^2}{N}} \approx \frac{\sqrt{2}\,\widehat{Var}(\hat\theta)}{\sqrt{N}}$

b. Best is $\hat{E}\{\widehat{Var}(\hat\theta)\} / \widehat{Var}(\hat\theta)$

but difference is acceptable