Solutions Midterm I

1 a). $L(\lambda_1, \lambda_2; (\delta_{ij}, U_{ij})) = \prod_{i=1}^{2} \prod_{j=1}^{n_i} \left(\lambda_i e^{-\lambda_i U_{ij}}\right)^{\delta_{ij}} \left(e^{-\lambda_i U_{ij}}\right)^{1-\delta_{ij}}$

b) $\ell(\lambda_1, \lambda_2) = \sum_{i=1}^{2} \sum_{j=1}^{n_i} \left(\delta_{ij} \log \lambda_i - \lambda_i U_{ij}\right)$

$$S(\lambda_1, \lambda_2) = \left(\frac{\partial \ell(\lambda_1 \lambda_2)}{\partial \lambda_1}, \frac{\partial \ell(\lambda_1 \lambda_2)}{\partial \lambda_2}\right)$$
$$= \left(\frac{\sum_j \delta_{1j}}{\lambda_1} - \sum_j U_{1j}, \frac{\sum \delta_{2j}}{\lambda_2} - \sum_j U_{2j}\right)$$

c) $S(\hat{\lambda}_1, \hat{\lambda}_2) = 0 \iff \hat{\lambda}_1 = \frac{\sum \delta_{1j}}{\sum U_{1j}}$; $\hat{\lambda}_2 = \frac{\sum \delta_{2j}}{\sum U_{2j}}$

$$\frac{\partial^2 \ell(\lambda_1 \lambda_2)}{\partial(\lambda_1 \lambda_2)^2} = \begin{pmatrix} -\frac{1}{\lambda_1^2} \sum \delta_{1j} & 0 \\ 0 & -\frac{1}{\lambda_2^2} \sum \delta_{2j} \end{pmatrix}$$
; $-\ell''(\lambda_1 \lambda_2)$ is pd $\Rightarrow \hat{\lambda}_1, \hat{\lambda}_2$ pt max bc $\frac{\partial^2 \ell(\lambda_1 \lambda_2)}{\partial \lambda_1 \partial \lambda_2} = 0$

d) $I_T(\lambda_1 \lambda_2) = \text{diag}\left(\frac{n_1}{\lambda_1^2} E[\delta_{1j}], \frac{n_2}{\lambda_2^2} E[\delta_{2j}]\right) = \text{diag}\left(\frac{n_1}{\lambda_1^2}(1 - e^{-\lambda_1 C_1}), \frac{n_2}{\lambda_2^2}(1 - e^{-\lambda_2 C_2})\right)$

since $E[\delta_{1j}] = P(\delta_{1j} = 1) = P(T_{1j} \le C_1) = 1 - e^{-\lambda_1 C_1}$.

e) $H_0: \lambda_1 = \lambda_2$ vs $H_a: \lambda_1 \neq \lambda_2$ - Take $h(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2$ $\nabla h(\lambda_1, \lambda_2) = (1, -1)$

$$T_W = h(\hat{\lambda}_1 \hat{\lambda}_2) \left(\nabla h(\hat{\lambda}_1 \hat{\lambda}_2) I_T^{-1}(\hat{\lambda}_1 \hat{\lambda}_2) \nabla h^T(\hat{\lambda}_1 \hat{\lambda}_2)\right)^{-1} h(\hat{\lambda}_1 \hat{\lambda}_2) = (\hat{\lambda}_1 - \hat{\lambda}_2)^2 \left\{(1 \;\; -1) \begin{pmatrix} \frac{\hat{\lambda}_1^2}{n_1} \hat{p}_1 & 0 \\ 0 & \frac{\hat{\lambda}_2^2}{n_2} \hat{p}_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}^{-1}$$

$$= \frac{(\hat{\lambda}_1 - \hat{\lambda}_2)^2}{\frac{\hat{\lambda}_1^2}{n_1} \hat{p}_1 + \frac{\hat{\lambda}_2^2}{n_2} \hat{p}_2}$$
when $\hat{p}_1 = 1 - e^{-\hat{\lambda}_1 C_1}$, $\hat{p}_2 = 1 - e^{-\hat{\lambda}_2 C_2}$

f) Under H_0, $T_W \sim \chi_1^2$.

2 a) $L(p_1, p_2, y_{ij}) = \prod_{i=1}^{2} \prod_{j=1}^{n_i} p_i^{y_{ij}} (1 - p_i)^{1 - y_{ij}}$

$$\ell(p_1 p_2) = \sum_i \sum_j \{y_{ij} \log p_i + (1 - y_{ij}) \log(1 - p_i)\}.$$

b) $S(p_1, p_2) = (\ell_{p_1}(p_1 p_2), \ell_{p_2}(p_1 p_2)) = \left(\frac{1}{p_1} \sum Y_{1j} - \frac{1}{1-p_1}(n_1 - \sum Y_{1j}), \frac{1}{p_2} \sum Y_{2j} - \frac{1}{1-p_2}(n_2 - \sum Y_{2j})\right)$

c) $i_T(p_1 p_2) = E\left[-\frac{\partial^2 \ell(p_1 p_2)}{\partial(p_1 p_2)^2}\right] = \text{diag}\left(\frac{1}{p_1^2} n_1 E Y_{1j} + \frac{n_1}{(1-p_1)^2}(1 - E Y_{1j}), \frac{n_2}{p_2^2} E Y_{2j} + \frac{n_2}{(1-p_2)^2}(1 - E Y_{2j})\right)$

bc $\frac{\partial^2 \ell(p_1 p_2)}{\partial p_1 \partial p_2} = 0 \Rightarrow i_T(p_1 p_2) = \text{diag}\left(\frac{n_1}{p_1} + \frac{n_1}{1-p_1}, \frac{n_2}{p_2} + \frac{n_2}{1-p_2}\right)$

$$= \text{diag}\left(\frac{n_1}{p_1(1-p_1)}, \frac{n_2}{p_2(1-p_2)}\right)$$

d) $T_S = \mathcal{I}(½,½)^T\, \mathcal{I}_T^{-1}(½,½)\, S(Y_2,½)$ $\quad H_0: P_1 = ½,\ P_2 = ½$ vs $H_a: P_1 \neq ½$ or $P_2 \neq ½$

$S(½,½) = \left(2Y_{1\cdot} - 2(n_1 - Y_{1\cdot}),\ 2Y_{2\cdot} - 2(n_2 - Y_{2\cdot})\right) = \left(4Y_{1\cdot} - 2n_1,\ 4Y_{2\cdot} - 2n_2\right)$

$\mathcal{I}_T^{-1}(½,½) = \mathrm{diag}\left(1/4n_1,\ 1/4n_2\right)$

$T_S = \frac{1}{4n_1}(4Y_{1\cdot} - 2n_1)^2 + \frac{1}{4n_2}(4Y_{2\cdot} - 2n_2)^2 \qquad Y_{1\cdot} = \sum_j Y_{1j};\ Y_{2\cdot} = \sum_j Y_{2j}$

e) $T_S \sim \chi^2_2$ under H_0 $(P_1 = ½,\ P_2 = ½)$

$\boxed{3}$ $Y_i \sim \frac{1}{\lambda}\exp\left(-\frac{y-\theta}{\lambda}\right),\ y \geq \phi$

a) $L(\phi,\lambda) = \left(\frac{1}{\lambda}\right)^n \exp\left\{-\frac{1}{\lambda}\sum(Y_i - \phi)\right\} \mathbb{1}(Y_{(1)} \geq \phi) \qquad Y_{(1)} = \min(Y_1, \ldots Y_n)$

$\hat\phi_{MLE} = Y_{(1)}$ is the maximizer of $L(\phi,\lambda)$ wrt ϕ.

For fixed ϕ $\Rightarrow$ the max of $L(\phi,\lambda)$ wrt λ can be found by $\frac{d}{d\lambda}\log L(\phi,\lambda) = 0$

$\Leftrightarrow \frac{d}{d\lambda}\left(-n\log\lambda - \frac{1}{\lambda}\sum(Y_i - \phi)\right) = 0 \Leftrightarrow -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum(Y_i - \phi) = 0 \Rightarrow \hat\lambda_{MLE} = \bar Y - \hat\phi_{MLE}$

since $\hat\lambda_\phi = \bar Y - \phi$ max the likel for fixed ϕ.

b) First calculate the dist'n of $\hat\phi = Y_{(1)}$: $P(Y_{(1)} \geq y) = P(Y_1 \geq y, \ldots Y_n \geq y) = \prod P(Y_i \geq y)$

$P(Y_{(1)} \geq y) = \exp\left(-\frac{n(y-\phi)}{\lambda}\right) \Rightarrow \hat\phi_{MLE} \sim$ Exp with loc ϕ, scale $= \lambda/n$.

or $n(\hat\phi_{MLE} - \phi) \sim \mathrm{Exp}(\lambda)$. λ - is the scale (mean) of exponential dist'n.

$\Rightarrow \hat\phi_{MLE} = \phi + O_p(n^{-1})$

$\hat\lambda_{MLE} = \bar Y - \hat\phi_{MLE}$ #

From CLT we have $\sqrt{n}(\bar Y - \phi - \lambda) \xrightarrow{d} N(0,\lambda^2)$; $\mathrm{Exp}(\phi,\lambda)$ has mean $= \phi + \lambda$ and variance λ^2

$\Rightarrow \sqrt{n}(\hat\lambda_{MLE} - \lambda) = \underbrace{\sqrt{n}(\bar Y - \phi - \lambda)}_{\xrightarrow{d} N(0,\lambda^2)} + \underbrace{\sqrt{n}(\hat\phi_{MLE} - \phi)}_{\xrightarrow{p} 0} \xrightarrow[\text{slutsky's thm}]{d} N(0,\lambda^2)$.

$\Rightarrow \hat\lambda_{MLE} = \lambda + O_p(n^{-1/2})$

c) Let $s,t \in \mathbb{R}$

Note $(s\hat\phi_{MLE} + t\hat\lambda_{MLE} - s\phi - t\lambda)\sqrt{n} = \underbrace{s\sqrt{n}(\hat\phi_{MLE} - \phi)}_{\xrightarrow{p} 0} + \underbrace{t\sqrt{n}(\hat\lambda_{MLE} - \lambda)}_{\xrightarrow{d} N(0,t^2\lambda^2)}$

$\Rightarrow (\hat\phi_{MLE}, \hat\lambda_{MLE})^T \sim AN_2\left(\begin{pmatrix}\phi\\ \lambda\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & \frac{1}{n}\lambda^2\end{pmatrix}\right)$

$\boxed{4}$ Reasons:
1) null value of ϕ on the boundary of param space
2) param space is countable (not ~~inter~~ compact set) $\Rightarrow$ no derivatives exist.
3) Under H_0, β is not identifiable
4) Null value of ϕ depends on n; thus asy $(n\to\infty)$ doesn't make sense.

$\boxed{5}$ a) No counterexample $X = X_n \sim N(0,1)$; $Y = Y_n \sim N(0,1)$. That are indep

$\Rightarrow X_n - Y_n \sim N(0,2)$ (thus $X_n - Y_n \not\xrightarrow{d} 0$)

b) $S(\underset{\sim}{\theta}, Y) \sim AN_k\left(0_k,\ \mathcal{I}_T(\theta)\right) \qquad \mathcal{I}_T(\theta) =$ total Fisher Info.