5.2 — Stochastic Convergence for scolar rand var. (5.2, 5.3

Review. scolar rand var

Let Y_n sequence of rand variables.

Conv almost surely (wp1)

$$Y_n \xrightarrow{as} Y \quad \text{if} \quad P\left(\lim_{n\to\infty} Y_n = Y\right) = 1$$

Techicol: $P(\forall \varepsilon \; \exists n_\varepsilon \; |Y_n - Y| < \varepsilon \;\; \forall n \geq n_\varepsilon) =$

Conv in probab

$$Y_n \xrightarrow[n\to\infty]{P} Y \quad \text{if} \quad \forall \varepsilon > 0 \;\; \lim_{n\to\infty} P\left(|Y_n - Y| > \varepsilon\right) = 0$$

Conv in dist'n

$Y_n \xrightarrow[n\to\infty]{d} Y$ (with dist'n fn F) if

$P(Y_n \leq y) \xrightarrow[n\to\infty]{} P(Y \leq y)$ $\forall y$ a point of continuity of F

Relationships between different types of convergence:

- Conv almost surely $\Longrightarrow$ Conv in probab.
  Converse NOT true.
  exemple: $Y_n \sim$ indep Bernoulli $\left(\frac{1}{n}\right)$

- Conv in probab $\Longrightarrow$ Conv in dist'n
  Converse true if the limiting rand var is NOT randa
  $Y_n \sim$ indep $N(\mu, \sigma_n^2)$ with $\sigma_n^2 \to \sigma^2$ $\Rightarrow Y_n \xrightarrow{d} N(\mu, \sigma^2)$

Review : Important results about stochastic convergence.

$\boxed{\text{Markov ineq}}$ X is rv and $E|X| < \infty$ Then

$$P(|X| > \alpha) \le \frac{EX}{\alpha} \quad \forall \alpha > 0.$$

Interpretation: upper bound on the probab that $|X|$ is greater or equal than pos number, when its expectation is finite.

$\boxed{\text{Chebyshev's ineq}}$ X is rv s.t $EX^2 < \infty$ Then

$$P(|X - EX| > \alpha) \le \frac{\text{Var}X}{\alpha^2} \quad \forall \alpha > 0$$

By strengthening the assn about the dist'n (2nd mom is finite) then we can control the deviation from its mean.

There are 2 immediate consequences (apply to sample mean)

1. $X_1 \dots X_n \sim \text{IID}$ $E|X_1| < \infty$

$$P(|\bar{X}| > \alpha) \le \frac{EX_1}{\alpha} \qquad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

2. $X_1 \dots X_n \sim \text{IID}$ $EX_1^2 < \infty$

$$P(|\bar{X} - EX_1| > \alpha) \le \frac{\text{Var}X_1}{n\alpha^2}$$

Thus by assuming $EX_1^2 < \infty \Rightarrow$ we can control the improvement in estimating the mean using the sample mean as $n \nearrow$.

Fundamental results in Statistics

$\boxed{\text{LLN}}$

$Y_1 \dots Y_n \sim IID \quad EY_1 < \infty$

WLLN: $\bar{Y} \xrightarrow{P} EY_1$ as $n \to \infty$

SLLN: $\bar{Y} \xrightarrow{as} EY_1$ as $n \to \infty$.

Clearly SLLN $\Rightarrow$ WLLN ; but SLLN is much more difficult to prove. You'll discuss both in ST779, next seme

Remarks Identically distributed can be replaced by uniformly integrable.

1. Both versions hold for pairwise indep (as opposed to just indep)

2. If the 2nd mom is assumed finite $\Rightarrow$ the proof follows easily from Chebyshev's ineq to $\frac{1}{n}\Sigma Y_i$

3 WLLN can be extended to accommodate sequence of RV not indep, not identically distributed.

Y_i are such $EY_i = \mu_i \quad \text{cov}(Y_i, Y_j) = \sigma_{ij}$ and

$\text{Var}\,\bar{Y} \to 0$. Then

$\frac{1}{n}\Sigma Y_i - \frac{1}{n}\Sigma \mu_i \xrightarrow{P} 0$

(proof straight forward).

$\boxed{\text{CLT}}$

$Y_1 \dots Y_n \sim IID \quad EY_i = 0 \quad EY_1^2 = 1$ Then

$\sqrt{n}\,\bar{Y} \xrightarrow{d} N(0,1)$ as $n \to \infty$.

Add discussion to contrast LLN (assumes finite mean) to CL

Thus stronger assn, stronger result!
We often write the result as $\frac{1}{n}\sum Y_i \sim AN(\mu, \frac{1}{n}\sigma^2)$, using the concept of "asymptotically normal" and pointing how $\frac{1}{n}\sum Y_i$ approaches μ.

(2) CLT implies that probabilistic and statistical methods that work for normal dist'n can be applied to many problems involving other dist'n (by working with the empirical mean).

(3) CLT refers to the behavior of the sample mean in a random sample. There is no analogue for the sample variance!
The behavior of the sample variance can be derived by using CLT.

(4) While WLLN (and law of large numbers) refer to convergences "in probability" or "almost sure" and thus don't require a multivariate version; this is not the case for CLT. We saw that convergence in dist'n of random vector s more that convergence in distribution of their components!