LR test $\quad T_{LR} = 2\ell(\hat\theta_{MLE}) - 2\ell(\tilde\theta)$ as before.

Remarks
- T_{LR}, T_S are para invariant and invariant to the choice of function $h(\theta)$.
- T_W is double para variant. Depends on the choice of para for θ and choice of $h(\theta)$
- Under H_0, it is expected the various versions of T_W to be similar.

Illustration: Behrens-Fisher problem

$X_1 \dots X_{n_1}$ NIID $N(\mu_1, \sigma_1^2)$ $\qquad Y_1, \dots, Y_{n_2}$ NIID $N(\mu_2, \sigma_2^2)$

Goal test $H_0: \mu_1 = \mu_2$

Sol: $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$

$h(\theta) = \mu_1 - \mu_2$

$H(\theta) = \nabla h(\theta) = (1, -1, 0, 0)$

$\ell(\theta) = -\frac{n_1}{2}\log\sigma_1^2 - \frac{n_2}{2}\log\sigma_2^2 - \frac{1}{2\sigma_1^2}\sum(X_i-\mu_1)^2 - \frac{1}{2\sigma_2^2}\sum(Y_i-\mu_2)^2$

$\hat\theta_{MLE} = \left(\bar X, \bar Y, \frac{1}{n_1}\sum(X_i-\bar X)^2, \frac{1}{n_2}\sum(Y_i-\bar Y)^2\right)$

$\tilde\theta = \underset{\mu_1=\mu_2}{\operatorname{argmax}}\ \ell(\theta)$

Do: take partial derivatives, set them to zero solve for $(\mu, \sigma_1^2, \sigma_2^2)$

$$\begin{cases} \sigma_1^2 = \frac{1}{n_1}\sum(X_i-\mu)^2 \\ \sigma_2^2 = \frac{1}{n_2}\sum(Y_i-\mu)^2 \\ \mu = \dfrac{n_1\bar X/\sigma_1^2 + n_2\bar Y/\sigma_2^2}{n_1/\sigma_1^2 + n_2/\sigma_2^2} \end{cases}$$

$$I_T(\theta) = \text{diag}\left(\frac{n_1}{\sigma_1^2}, \frac{n_2}{\sigma_2^2}, \frac{n_1}{2\sigma_1^4}, \frac{n_2}{2\sigma_2^4}\right)$$

$$\nabla h(\theta) = \frac{d}{d\theta} h(\theta) = (1, -1, 0, 0)$$

$$T_W = \frac{(\bar{X}-\bar{Y})^2}{\left(\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}\right)}$$

usual t-test

$\boxed{\text{No need } \tilde{\theta}}$

To calculate T_S we need $S(\theta)$

$\boxed{\text{Need } \tilde{\theta}}$

$$S(\theta) = \begin{pmatrix} \frac{n_1}{\sigma_1^2}(\bar{X}-\mu_1) \\ \frac{n_2}{\sigma_2^2}(\bar{Y}-\mu_2) \\ -\frac{n_1}{\sigma_1^2} + \frac{1}{2\sigma_1^4}\Sigma(X-\mu_1)^2 \\ -\frac{n_2}{\sigma_2^2} + \frac{1}{2\sigma_2^4}\Sigma(Y-\mu_2)^2 \end{pmatrix}$$

$$S(\tilde{\theta}) = \begin{pmatrix} \frac{n_1}{\sigma_1^2}(\bar{X}-\mu) \\ \frac{n_2}{\sigma_2^2}(\bar{Y}-\mu) \\ 0 \\ 0 \end{pmatrix}$$

$$T_S = \frac{n_1^{\cancel{2}}}{\sigma_1^{\cancel{4}2}}(\bar{X}-\mu)^2 \cdot \frac{\cancel{\sigma_1^2}}{\cancel{n_1}} + \frac{n_2^{\cancel{2}}}{\sigma_2^{\cancel{4}2}}(\bar{Y}-\mu)^2 \frac{\cancel{\sigma_2^2}}{\cancel{n_2}}$$

$$T_S = \frac{n_1}{\sigma_1^2}(\bar{X}-\mu) + \frac{n_2}{\sigma_2^2}(\bar{Y}-\mu)^2$$

$$\bar{X} - \mu = \bar{X} - \frac{\frac{n_1}{\sigma_1^2}\bar{X} + \frac{n_2}{\sigma_2^2}\bar{Y}}{\frac{n_1}{\sigma_1^2} + \frac{n_2}{\sigma_2^2}} = \frac{\frac{n_2}{\sigma_2^2}(\bar{X}-\bar{Y})}{\frac{n_1}{\sigma_1^2} + \frac{n_2}{\sigma_2^2}}$$

$$\frac{n_1}{\sigma_1^2}(\bar{x}-\mu)^2 = \frac{n_1 n_2^2}{\cancel{\sigma_1^2}\,\cancel{\sigma_2^4}}(\bar{x}-\bar{y})^2 \frac{\overset{\sigma_1^2}{\cancel{\sigma_1^4}} \cdot \cancel{\sigma_2^4}}{(n_1\sigma_2^2 + n_2\sigma_1^2)^2}$$

$$\frac{n_2}{\sigma_2^2}(\bar{Y}-\mu)^2 = \frac{n_2 n_1^2}{\sigma_2^{-2}} \frac{(\bar{X}-\bar{Y})^2}{(n_1\sigma_2^2 + n_2\sigma_1^2)^2}.$$

$$T_S = (\bar{X}-\bar{Y})^2 \cdot \frac{1}{(n_1\sigma_2^2 + n_2\sigma_1^2)} \cancel{= n_1 n_2 (n_2\sigma_1^2 + n_1\sigma_2^2)}$$

$$T_S = \frac{(\bar{X}-\bar{Y})^2}{\frac{\tilde{\sigma}_1^2}{n_1} + \frac{\tilde{\sigma}_2^2}{n_2}}$$

T_{LR}: we need $\ell(\hat{\theta}_{MLE})$ and $\ell(\tilde{\theta})$ $\boxed{\text{Need } \tilde{\theta}}$

$$\ell(\hat{\theta}_{MLE}) = -\frac{n_1}{2}\log\hat{\sigma}_1^2 - \frac{n_1}{2} - \frac{n_2}{2}\log\hat{\sigma}_2^2 - \frac{n_2}{2}$$

$$\ell(\tilde{\theta}) = -\frac{n_1}{2}\log\tilde{\sigma}_1^2 - \frac{n_1}{2} - \frac{n_2}{2}\log\tilde{\sigma}_2^2 - \frac{n_2}{2}$$

$$T_{LR} = -n_1\log\frac{\hat{\sigma}_1^2}{\tilde{\sigma}_1^2} - n_2\log\frac{\hat{\sigma}_2^2}{\tilde{\sigma}_2^2}$$