Taylor series expansion of $g(Y_i; \hat{\theta})$ around $\theta$

$$g(Y_i; \hat{\theta}) = g(Y_i; \theta) + (\hat{\theta}-\theta)^T g_\theta(Y_i; \theta) + \frac{1}{2}(\hat{\theta}-\theta) g_{\theta\theta}(Y_i; \theta^*)(\hat{\theta}-\theta)$$

where $\theta^*$ is between $\theta$ and $\hat{\theta}$

This implies that

$$T_n = \frac{1}{n}\sum_{i=1}^n g(Y_i; \hat{\theta})$$

$$= \frac{1}{n}\sum_{i=1}^n g(Y_i; \theta) + (\hat{\theta}-\theta) \frac{1}{n}\sum_{i=1}^n g_\theta(Y_i; \theta)$$

$$+ \frac{1}{2}(\hat{\theta}-\theta)^T \frac{1}{n}\sum_{i=1}^n g_{\theta\theta^T}(Y_i; \theta_i^*) \quad (\hat{\theta}-\theta)$$

(Replace $\hat{\theta}-\theta$ by $\frac{1}{n}\sum h(Y_i) + R_n$ in 2nd summand)

$$= \frac{1}{n}\sum g(Y_i; \theta) \qquad (A1)$$

$$+ \left(\frac{1}{n}\sum_{i=1}^n h(Y_i) + R_n\right)\left(\frac{1}{n}\sum g_\theta(Y_i; \theta)\right) \qquad (A2)$$

$$+ \frac{1}{2}(\hat{\theta}-\theta)^T \frac{1}{n}\sum g_{\theta\theta^T}(Y_i; \theta_i^*) (\hat{\theta}-\theta) \qquad (A3)$$

$$= A1 + A2 + A3$$

where we examine next each term in turn. We start with A3 and slowly move towards A1.

$$A_3 = \frac{1}{2}\underbrace{(\hat\theta-\theta)^T}_{O_p(n^{-1/2})} \overbrace{\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} g_{\theta\theta^T}(Y_i;\theta_i^*)\right)}}^{b\times b \text{ matrix}} \underbrace{(\hat\theta-\theta)}_{O_p(n^{-1/2})}$$

the term in the middle has the summand with the property:

$$\left| g_{\theta_j\theta_{j'}}(Y_i,\theta_i^*) \right| \le M(Y_i) \quad \forall\ \theta_i^* \text{ around } \theta$$

with $EM(Y_1) < \infty$.

$$\left| \frac{1}{n}\sum g_{\theta_j\theta_{j'}}(Y_i,\theta_i^*) \right| \le \underbrace{\frac{1}{n}\sum M(Y_i)}_{= O_p(1) \text{ since } EM(Y_1)<\infty}$$

Thus $\underline{A_3 = O_p(n^{-1/2})\, O_p(1)\, O_p(n^{-1/2}) = o_p(n^{-1/2})}$

$$A_2 \quad \left(\frac{1}{n}\sum h(Y_i) + R_n\right)\left(\frac{1}{n}\sum g_\theta(Y_i;\theta)\right)$$

$$= \frac{1}{n}\sum h(Y_i)\ \frac{1}{n}\sum g_\theta(Y_i;\theta)$$

$$+ \underbrace{R_n}_{o_p(n^{-1/2})}\ \underbrace{\frac{1}{n}\sum g_\theta(Y_i;\theta)}_{O_p(1) \text{ since } E g_\theta(Y_1;\theta) < \infty}$$

$$= \left(\frac{1}{n}\sum_i h(Y_i)\right)\cdot \underbrace{\left(\frac{1}{n}\sum g_\theta(Y_i;\theta)\right)}_{\text{write this as } E g_\theta(Y_1;\theta) + \left\{\frac{1}{n}\sum g_\theta(Y_i;\theta) - E g_\theta(Y_1;\theta)\right\}} + o_p(n^{-1/2})$$

$$= \frac{1}{n} \sum_i h(Y_i) \cdot E\left[g_\theta(Y_i;\theta)\right]$$

$$+ \underbrace{\left(\frac{1}{n} \sum h(Y_i)\right)}_{O_p(n^{-1/2})} \underbrace{\left(\frac{1}{n} \sum g_\theta(Y_i,\theta) - E\, g_\theta(Y_i;\theta)\right)}_{o_p(1) \text{ from WLLN}} + o_p(n^{-1/2})$$

Thus

$$A_2 = \frac{1}{n} \sum_i h(Y_i)\, E\left(g_\theta(Y_i;\theta)\right) + o_p(n^{-1/2})$$

Since $\frac{1}{n} \sum g(Y_i,\theta) \xrightarrow{P} E\left[g(Y_i,\theta)\right]$ WLLN we can represent (*)

$$T_n = E\left[g(Y_i;\theta)\right] + \frac{1}{n} \sum_{i=1}^{n} \left\{ \underbrace{g(Y_i;\theta) - E\left[g(Y_i;\theta)\right] + h(Y_i)\, E\left[g_\theta(Y_i;\theta)\right]}_{h_T(Y_i)} \right\}$$

$$+ R_{2n} \qquad \text{where } R_{2n} = o_p(n^{-1/2}).$$

Since

$$E\left[g(Y_i,\theta) - E\left[g(Y_i,\theta)\right] + h(Y_i)\, E\left[g_\theta(Y_i;\theta)\right]\right]$$

$$= \underbrace{E\left[g(Y_i,\theta)\right] - E\, g(Y_i;\theta)}_{=0} + \underbrace{E\, Y(Y_i)}_{=0}\, E\left[g_\theta(Y_i,\theta)\right] = 0$$

$\Rightarrow$ the above representation (*) is the Bahadur repres of $T_n$ //.