# ST 793 : Solution of Homework-4

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## Book problems

### Problem 5.27

Let $Y_1, \ldots, Y_n$ are i.i.d with mean $\mu$ and variance $\sigma^2$. Define a vector valued random variable, $Z = (Y, Y^2)^\top$. Then, $Z_1, Z_2, \ldots, Z_n$ are i.i.d with mean $\boldsymbol{\mu}_z = (\mu, \sigma^2 + \mu^2)^\top$ and covariance matrix

$$\begin{aligned}
\boldsymbol{\Sigma} &= \begin{pmatrix} V(Y) & \mathrm{Cov}(Y, Y^2) \\ \mathrm{Cov}(Y, Y^2) & V(Y^2) \end{pmatrix} \\
&= \begin{pmatrix} \sigma^2 & \mu_3' - \mu(\sigma^2 + \mu^2) \\ \mu_3' - \mu(\sigma^2 + \mu^2) & \mu_4' - (\sigma^2 + \mu^2)^2 \end{pmatrix} \qquad \text{[In terms of raw moments]} \\
&= \begin{pmatrix} \sigma^2 & \mu_3 + 2\mu\sigma^2 \\ \mu_3 + 2\mu\sigma^2 & \mu_4 + 4\mu_3\mu + 4\mu^2\sigma^2 - \sigma^4 \end{pmatrix} \qquad \text{[In terms of central moments]}
\end{aligned}$$

Then, by central limit theorem,

$$\sqrt{n}\left[\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n Y_i \\ \\ \frac{1}{n}\sum_{i=1}^n Y_i^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \\ \sigma^2 + \mu^2 \end{pmatrix}\right] \xrightarrow{d} \mathrm{N}_2\left(\mathbf{0}, \boldsymbol{\Sigma}\right)$$

Now, define, a vector valued function $g : \mathbb{R}^2 \to \mathbb{R}^3$, such that,

$$g(\boldsymbol{\theta}) = g(\theta_1, \theta_2) = \left(\theta_1, \sqrt{\theta_2 - \theta_1^2}, \frac{\sqrt{\theta_2 - \theta_1^2}}{\theta_1}\right)^\top$$

Then, understand that,

$$g\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n Y_i \\ \\ \frac{1}{n}\sum_{i=1}^n Y_i^2 \end{pmatrix} = \left(\bar{Y}_n, s_n, \frac{s_n}{\bar{Y}_n}\right)^\top$$

Thus, by delta method,

$$\sqrt{n}\left[\begin{pmatrix} \bar{Y}_n \\ s_n \\ \frac{s_n}{\bar{Y}_n} \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma \\ \frac{\sigma}{\mu} \end{pmatrix}\right] \xrightarrow{d} \mathrm{N}_3\left(\mathbf{0}, g'(\boldsymbol{\mu}_\mathbf{z})\, \boldsymbol{\Sigma}\, g'(\boldsymbol{\mu}_\mathbf{z})^\top\right)$$

where,

$$g'(\boldsymbol{\theta}) = \frac{\partial g}{\partial \boldsymbol{\theta}} = \begin{pmatrix} 1 & -\frac{\theta_1}{\sqrt{\theta_2 - \theta_1^2}} & -\frac{\theta_2}{\theta_1^2\sqrt{\theta_2 - \theta_1^2}} \\ 0 & \frac{1}{2\sqrt{\theta_2 - \theta_1^2}} & \frac{1}{2\theta_1\sqrt{\theta_2 - \theta_1^2}} \end{pmatrix}^\top$$

Evaluating the above at $\boldsymbol{\theta} = (\mu, \sigma^2 + \mu^2)^\top$ we get

$$g'(\boldsymbol{\mu}_\mathbf{z}) = \begin{pmatrix} 1 & -\frac{\mu}{\sigma} & -\frac{1}{\sigma} - \frac{\sigma}{\mu^2} \\ 0 & \frac{1}{2\sigma} & \frac{1}{2\mu\sigma} \end{pmatrix}^\top$$

## Problem 5.28

Define, a vector valued function $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(\mathbf{x}) = (g_1(x_1), g_2(x_2))$. Then,

$$g'(\mathbf{x}) = \begin{pmatrix} g_1'(x_1) & 0 \\ 0 & g_2'(x_2) \end{pmatrix}$$

By the assumptions, $g'(\boldsymbol{\theta}) \neq 0$ as both $g_1'(\theta_1)$ and $g_2'(\theta_2)$ are non-zero. Then by delta method,

$$\sqrt{n}\left[\begin{pmatrix} g_1(\hat{\theta}_1) \\ \\ g_2(\hat{\theta}_2) \end{pmatrix} - \begin{pmatrix} g_1(\theta_1) \\ \\ g_2(\theta_2) \end{pmatrix}\right] \xrightarrow{d} N_2\left(\mathbf{0}, g'(\boldsymbol{\theta})\, \boldsymbol{\Sigma}\, g'(\boldsymbol{\theta})\right)$$

Because $g'(\mathbf{x})$ is a diagonal matrix we can do further simplification of the asymptotic covariance matrix, which is

$$g'(\boldsymbol{\theta})\, \boldsymbol{\Sigma}\, g'(\boldsymbol{\theta}) = \begin{pmatrix} (g_1'(\theta_1))^2 \sigma_{11} & g_1'(\theta_1)g_2'(\theta_2)\sigma_{12} \\ g_1'(\theta_1)g_2'(\theta_2)\sigma_{12} & (g_2'(\theta_2))^2 \sigma_{22} \end{pmatrix}$$

This means that the asymptotic correlation between $g_1(\hat{\theta}_1)$ and $g_2(\hat{\theta}_2)$ is given by

$$\begin{aligned} r(\hat{g}_1, \hat{g}_2) &= \frac{g_1'(\theta_1)g_2'(\theta_2)\sigma_{12}}{\sqrt{(g_1'(\theta_1))^2 \sigma_{11}\, (g_2'(\theta_2))^2 \sigma_{22}}} \\ &= \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \qquad [\text{because both } g_1'(\theta_1),\ \ g_2'(\theta_2) > 0 \text{ as } g_1, g_2 \text{ are increasing}] \\ &= r\left(\hat{\theta}_1, \hat{\theta}_2\right) \end{aligned}$$

This completes the proof.

## Problem 5.39

By Theorem 5.25 of the book,

$$\begin{aligned} \left(\hat{\eta}_{\frac{3}{4}} - \hat{\eta}_{\frac{1}{4}}\right) - \left(\eta_{\frac{3}{4}} - \eta_{\frac{1}{4}}\right) &= \frac{1}{n}\sum_{i=1}^{n}\left(\left[\frac{\frac{3}{4} - I(X_i \leq \eta_{\frac{3}{4}})}{F'\left(\eta_{\frac{3}{4}}\right)}\right] - \left[\frac{\frac{1}{4} - I(X_i \leq \eta_{\frac{1}{4}})}{F'\left(\eta_{\frac{1}{4}}\right)}\right]\right) + R_{1n} - R_{2n} \\ &= \frac{1}{n}\sum_{i=1}^{n} h_T(X_i) + R_n \end{aligned}$$

where

$$h_T(X_i) = \left[\frac{\frac{3}{4} - I(X_i \leq \eta_{\frac{3}{4}})}{F'\left(\eta_{\frac{3}{4}}\right)}\right] - \left[\frac{\frac{1}{4} - I(X_i \leq \eta_{\frac{1}{4}})}{F'\left(\eta_{\frac{1}{4}}\right)}\right] \qquad i = 1, 2, \ldots, n$$

$R_n = R_{1n} - R_{2n}$ and $\sqrt{n}R_n \to 0$ as $n \to \infty$ because both $\sqrt{n}R_{1n} \to 0$ and $\sqrt{n}R_{2n} \to 0$ as $n \to \infty$

## Problem 5.49

Let's denote $M(\mathbf{t})$ be the moment generating function (MGF) of a random variable. Then, $\mathbf{X}_n \to \mathbf{X}$ and $\mathbf{Y}_n \to \mathbf{Y}$ implies $M_{\mathbf{X}_n}(\mathbf{t}) \to M_{\mathbf{X}}(\mathbf{t})$ and $M_{\mathbf{Y}_n}(\mathbf{s}) \to M_{\mathbf{Y}}(\mathbf{s})$ for all $(\mathbf{s}, \mathbf{t})$. Then the moment generating function of $\mathbf{Z}_n = \mathbf{X}_n + \mathbf{Y}_n$ is given by

$$\begin{aligned} M_{\mathbf{Z}_n}(\mathbf{t}, \mathbf{s}) &= \mathrm{E}\left(\exp\left(\mathbf{t}'\mathbf{X}_n + \mathbf{s}'\mathbf{Y}_n\right)\right) \\ &= \mathrm{E}\left(\exp\left(\mathbf{t}'\mathbf{X}_n\right)\right)\mathrm{E}\left(\exp\left(\mathbf{s}'\mathbf{Y}_n\right)\right) \qquad [\mathbf{X}_n \text{ and } \mathbf{Y}_n \text{ are independent}] \\ &= M_{\mathbf{X}_n}(\mathbf{t})M_{\mathbf{Y}_n}(\mathbf{s}) \to M_{\mathbf{X}}(\mathbf{t})M_{\mathbf{Y}}(\mathbf{s}) = \mathrm{E}\left(\exp\left(\mathbf{t}'\mathbf{X} + \mathbf{s}'\mathbf{Y}\right)\right) \end{aligned}$$

The last step follows because $\mathbf{X}$ and $\mathbf{Y}$ are also independent. This proves that $\mathbf{X}_n + \mathbf{Y}_n$ converges to distribution to $\mathbf{X} + \mathbf{Y}$.

## Problem 5.52

$$
\begin{aligned}
v_{n,i} = \mathrm{Var}\left(\sum_{j\neq i} h_{ij}e_j\right) &= \sum_{j\neq i} h_{ij}^2 \mathrm{Var}\left(e_j\right) = \sigma^2 \sum_{j\neq i} h_{ij}^2 \\
&= \sigma^2\left(\sum_{j=1}^{n} h_{ij}^2 - h_{ii}^2\right) = \sigma^2\left(h_{ii} - h_{ii}^2\right) \qquad \text{because } \mathbf{H}^2 = \mathbf{H} \\
&= \sigma^2 h_{ii}\left(1 - h_{ii}\right)
\end{aligned}
$$

Now, notice that,

$$
\begin{aligned}
Y_i - \hat{Y}_i &= \left(\mathbf{Y} - \hat{\mathbf{Y}}\right)_i \\
&= \left((\mathbf{I}-\mathbf{H})\mathbf{Y}\right)_i = \left((\mathbf{I}-\mathbf{H})\mathbf{e}\right)_i \qquad [\text{because } (\mathbf{I}-\mathbf{H})\mathbf{X} = 0] \\
&= e_i - \sum_{j=1}^{n} h_{ij}e_j \\
&= (1-h_{ii})e_i - \sum_{j\neq i} h_{ij}e_j
\end{aligned}
$$

Define, $X_{n,j} = \sqrt{v_{n,i}} h_{ij}e_j \;\; j = 1,2,\ldots,n, j\neq i$ and $\alpha_{n,i} = \max_{j\neq i} \; |h_{ij}|$

Then, the Lindeberg condition asks us to prove that

$$
\begin{aligned}
\epsilon_n &= \sum_{\substack{j=1\\ j\neq i}}^{n} \mathrm{E}\left(X_{n,j}^2 I\left(|X_{n,j}| > \delta\right)\right) \\
&= v_{n,i}^{-2} \;\; \sum_{\substack{j=1\\ j\neq i}}^{n} \mathrm{E}\left(h_{ij}^2 e_j^2 I\left(|h_{ij}e_j| > \delta\right)\right) \\
&= v_{n,i}^{-1} \;\; \sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2 \mathrm{E}\left(e_j^2 I\left(|e_j| > \frac{\delta}{|h_{ij}|}\right)\right) \\
&\leq v_{n,i}^{-1} \;\; \sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2 \mathrm{E}\left(e_j^2 I\left(|e_j| > \frac{\delta}{\alpha_{n,i}}\right)\right) \\
&= v_{n,i}^{-1} \;\; \mathrm{E}\left(e_1^2 I\left(|e_1| > \frac{\delta}{\alpha_{n,i}}\right)\right) \sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}^2 \\
&= \sigma^{-2} \;\; \mathrm{E}\left(e_1^2 I\left(|e_1| > \frac{\delta}{\alpha_{n,i}}\right)\right) \to 0 \quad [\text{ by Dominated Convergence Theorem as } \alpha_{n,i} \to 0 \text{ as } n \to \infty]
\end{aligned}
$$

By Lindeberg Central Limit theorem,

$$
\sum_{\substack{j=1\\ j\neq i}}^{n} X_{n,j} = \sqrt{v_{n,i}} \sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}e_j \to \mathrm{N}(0,1)
$$

Also, because, $h_{ii} \to c_i$,

$$
\sqrt{v_{n,i}} \to \left(c_i(1-c_i)\right)^{1/2}
$$

By Slutsky's theorem,

$$
\sum_{\substack{j=1\\ j\neq i}}^{n} h_{ij}e_j \to \mathrm{N}\left(0, c_i(1-c_i)\right)
$$

This completes the proof.

As an alternative approach to check the lindeberg condition we can check,

$$\lim_{n\to\infty} \max_{j\neq i} P\left(|X_{n,i}| > \epsilon\right) = 0$$

which can be checked by applying Chebyshev's inequality because,

$$P\left(|X_{n,j}| > \epsilon\right) < \frac{\text{Var}(X_{n,j})}{\epsilon^2} = \frac{v_{n,i} h_{ij}^2 \sigma^2}{\epsilon^2}$$

This implies,

$$\max_{j\neq i} P\left(|X_{n,i}| > \epsilon\right) < \frac{v_{n,i}\sigma^2}{\epsilon^2} \max_{j\neq i} h_{ij}^2 \to 0$$

# Problem 2

Because $Y_1, Y_2, \ldots, Y_n$ follows $N(0, \sigma^2)$, $E(Y_i^2) = \sigma^2$ and $Var(Y_i^2) = E(Y_i^4) - E^2(Y_i^2). = 3\sigma^4 - \sigma^4 = 2\sigma^4$. Then, by central limit theorem,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \sigma^2\right) \xrightarrow{d} N\left(0, 2\sigma^4\right)$$

This means, $S_n^2 \sim \text{AN}\left(\sigma^2, \frac{2\sigma^4}{n}\right)$.

For a function $g(x)$ such that $g'(\sigma^2) \neq 0$, by delta method,

$$\sqrt{n}\left(g\left(S_n^2\right) - g\left(\sigma^2\right)\right) \xrightarrow{d} N\left(0, 2\left(g'(\sigma^2)\right)^2 \sigma^4\right)$$

We want a $g$ such that the asymptotic variance or standard deviation does not depend on $\sigma^2$, which means we want

$$g'(\sigma^2)\sigma^2 = c \qquad \text{for some constant } c > 0$$

which is equivalent to saying that $g$ must satisfy the differential equation

$$g'(x) = \frac{c}{x} \quad \forall \; x$$

which implies $g(x) = c\log(x)$ for some $c > 0$