Theorem $Y_1 \dots Y_n \sim IID\ f(y;\theta)$ which satisfies the regularity assn. And. $I(\theta)$ is cont

$T_{LR} = -2\{\ell(\tilde{\theta}) - \ell(\hat{\theta})\}$

Under H_0 $\theta_1 = \theta_{10}$, $T_{LR} \xrightarrow{d} \chi^2_r$ as $n \to \infty$

proof

$$\ell(\tilde{\theta}) = \ell(\hat{\theta}) + (\tilde{\theta} - \hat{\theta})' \underbrace{\ell_\theta(\hat{\theta})}_{=0} + \frac{1}{2}(\tilde{\theta} - \hat{\theta})^T \ell_{\theta\theta^T}(\hat{\theta}^*)(\tilde{\theta} - \hat{\theta})$$

$$-2\{\ell(\tilde{\theta}) - \ell(\hat{\theta})\} = (\tilde{\theta} - \hat{\theta})^T \{-\ell_{\theta\theta^T}(\hat{\theta}^*)\}(\tilde{\theta} - \hat{\theta})$$

Road map. 1) find asy dist'n of $(\tilde{\theta} - \hat{\theta})\sqrt{n}$

2) find limit in prob of $-\ell_{\theta\theta^T}(\hat{\theta}^*)\frac{1}{n}$

We start with 2) $\ell_{\theta\theta^T}(\hat{\theta}^*) = \ell_{\theta\theta^T}(\theta_0) + \sum_{j=1}^{r} \frac{\partial^3 \ell(\hat{\theta}^{**})}{\partial\theta\partial\theta^T\partial\theta_j}(\hat{\theta}_j^* - \theta_{0j})$

$$\left| -\frac{1}{n}\ell_{\theta\theta^T}(\hat{\theta}^*) + \frac{1}{n}\ell_{\theta\theta^T}(\theta_0) \right| \leq \underbrace{\frac{1}{n}\sum h(Y_i)}_{O_p(1)} \cdot \underbrace{\sum_{j=1}^{r}(\hat{\theta}_j^* - \theta_{0j})}_{o_p(1)}$$

$$\Rightarrow -\frac{1}{n}\ell_{\theta\theta^T}(\hat{\theta}^*) \xrightarrow{P} I(\theta_0)$$

since $-\frac{1}{n}\ell_{\theta\theta^T}(\theta_0) \xrightarrow{P} I(\theta_0)$

To prove 1) ; recall

$$\hat{\theta} - \theta_0 = \frac{1}{n}\sum I^{-1}(\theta_0)\frac{d}{d\theta}\log f(Y_i;\theta_0) + R_n$$

$$\tilde{\theta}_2 - \theta_{20} = \frac{1}{n}\sum I_{22}^{-1}(\theta_0)\frac{\partial}{\partial\theta_2}\log f(Y_i;\theta_0) + Q_n.$$

How, we want $\tilde{\theta} - \theta_0$; for this we make up a matrix $H(\theta)$ by padding $I_{22}^{-1}(\theta_0)$ with 0-block matrices

$$\underset{b\times b}{H(\theta)} \triangleq \begin{pmatrix} 0 & 0 \\ 0 & I_{22}^{-1}(\theta_0) \end{pmatrix} \quad ; \quad \underset{b\times 1}{\tilde{Q}_n} = \begin{pmatrix} 0 \\ Q_n \end{pmatrix}$$

$$\tilde{Q}_n = o_p(n^{-1/2})$$

and obtain

$$\tilde{\theta} - \theta_0 = \frac{1}{n} \sum_{i=1}^{n} H(\theta_0) \frac{\partial}{\partial \theta} \log f(Y_i; \theta) + \tilde{Q}_n$$

$$\tilde{\theta} - \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \left\{ H(\theta_0) - I^{-1}(\theta_0) \right\} \frac{\partial}{\partial \theta} \log f(Y_i; \theta_0) + o_p(n^{-1/2})$$

$$\operatorname{cov}\left(\left\{ H(\theta_0) - I^{-1}(\theta_0) \right\} \frac{\partial}{\partial \theta} \log f(Y_i; \theta_0) \right)$$

$$= \left\{ H(\theta_0) - I^{-1}(\theta_0) \right\} . I(\theta_0) \left\{ H(\theta_0) - I^{-1}(\theta_0) \right\}^T$$

$$= H(\theta_0) I(\theta_0) H^T(\theta_0) - H(\theta_0) - H(\theta_0) + I^{-1}(\theta_0) = I^{-1}(\theta_0) - H(\theta_0)$$

$$\underbrace{H(\theta_0) I(\theta_0) H^T(\theta_0)}_{= H(\theta_0)}$$

$$\Rightarrow \sqrt{n}(\tilde{\theta} - \hat{\theta}) \xrightarrow{d} N_b\left(0, I^{-1}(\theta_0) - H(\theta_0)\right)$$

Note the asy variance is of dim $b \times b$. $I^{-1}(\theta_0)\left\{ I_{bb} - I(\theta_0) H(\theta_0) \right\}$

However we have result

$$= I^{-1}(\theta_0) \begin{pmatrix} I_{rr} & -I_{12} I_{22}^{-1} \\ 0 & 0 \end{pmatrix}$$

> $X_n \xrightarrow{d} N(0, \Sigma)$ and $D_n \xrightarrow{P} D$ then
> if $D\Sigma$ is idempotent and $r = \operatorname{rank} D\Sigma = \operatorname{Trace} D\Sigma$
> $X_n^T D_n X_n \xrightarrow{d} \chi_r^2$

Use this result for $X_n = \sqrt{n}(\tilde{\theta} - \hat{\theta})$; $\Sigma = I^{-1}(\theta_0) - H(\theta_0)$; $D = I(\theta_0)$ after checking that $D\Sigma = I_{b\times b} - I(\theta_0) H(\theta_0)$ which is idemp and $\operatorname{Trace} D\Sigma = b - (b-r) = r \Rightarrow n(\tilde{\theta} - \hat{\theta})^T \{ -\ell \ldots \} (\tilde{\theta} - \hat{\theta}) \xrightarrow{d} \chi_r^2$ //