ST 793 : Solution of Homework-5

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Book problems

Problem 5.10

$$\begin{aligned}\mathbf{Y}_n^\top \mathbf{C}_n \mathbf{Y}_n - \mathbf{Y}^\top \mathbf{C}\mathbf{Y} &= \mathbf{Y}_n^\top \mathbf{C}_n \mathbf{Y}_n - \mathbf{Y}_n^\top \mathbf{C}\mathbf{Y}_n + \mathbf{Y}_n^\top \mathbf{C}\mathbf{Y}_n - \mathbf{Y}^\top \mathbf{C}\mathbf{Y} \\ &= \mathbf{Y}_n^\top \left(\mathbf{C}_n - \mathbf{C}\right) \mathbf{Y}_n + \mathbf{Y}_n^\top \mathbf{C}\mathbf{Y}_n - \mathbf{Y}^\top \mathbf{C}\mathbf{Y} \end{aligned} \tag{1}$$

Now, $\mathbf{C}_n \xrightarrow{p} \mathbf{C}$ implies $\mathbf{C}_n - \mathbf{C} \xrightarrow{p} 0$. This along with $\mathbf{Y}_n \xrightarrow{d} \mathbf{Y}$ implies that by Slutsky's theorem,

$$\mathbf{Y}_n^\top \left(\mathbf{C}_n - \mathbf{C}\right) \mathbf{Y} \xrightarrow{d} \mathbf{Y}^\top \mathbf{0} \mathbf{Y} = 0$$

Because, convergence in distribution to a constant implies convergence in probability, we have

$$\mathbf{Y}_n^\top \left(\mathbf{C}_n - \mathbf{C}\right) \mathbf{Y} \xrightarrow{p} 0 \tag{2}$$

For the second part in (1) we will use continuous mapping theorem. Take $g : \mathbb{R}^k \to \mathbb{R}$ such that $g(\mathbf{X}) = \mathbf{X}^\top \mathbf{C}\mathbf{X}$. Then g is a continuous function of $\mathbf{X}$. By CMT, we have

$$\mathbf{Y}_n^\top \mathbf{C}\mathbf{Y}_n = g(\mathbf{Y}_n) \xrightarrow{d} g(\mathbf{Y}) = \mathbf{Y}^\top \mathbf{C}\mathbf{Y} \tag{3}$$

Equation (2) and (3) together completes the proof.

Problem 6.9

The Wald test-statistic is

$$T_w = n\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_{10}\right)^\top \left(\left[\mathbf{I}(\hat{\boldsymbol{\theta}})^{-1}\right]_{11}\right)^{-1} \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_{10}\right)$$

Because, $\mathbf{I}(\boldsymbol{\theta})$ is a continuous function of $\boldsymbol{\theta}$ and the fact that $\hat{\boldsymbol{\theta}}_1 \xrightarrow{p} \boldsymbol{\theta}_{10}$, we have

$$\left[\mathbf{I}(\hat{\boldsymbol{\theta}})^{-1}\right]_{11} \xrightarrow{p} \left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11} \tag{4}$$

From equation (6.22) in the book, we get

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_{10}\right) \xrightarrow{d} \mathrm{N}\left(\mathbf{d}, \left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11}\right) \qquad \text{[Under the Pitman alternatives]} \tag{5}$$

Let $\mathbf{Y}_n = \sqrt{n}\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_{10}\right)$, $\mathbf{C}_n = \left(\left[\mathbf{I}(\hat{\boldsymbol{\theta}})^{-1}\right]_{11}\right)^{-1}$ and $\mathbf{C} = \left(\left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11}\right)^{-1}$ Then, using (4), (5) and the result of the problem 5.10 we have

$$T_w = n\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_{10}\right)^\top \left(\left[\mathbf{I}(\hat{\boldsymbol{\theta}})^{-1}\right]_{11}\right)^{-1} \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_{10}\right) \xrightarrow{d} \mathbf{Y}^\top \left(\left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11}\right)^{-1} \mathbf{Y}$$

where $\mathbf{Y}$ follows $\mathrm{N}\left(\mathbf{d}, \left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11}\right)$. So, all we have to find the distribution of $\mathbf{Y}^\top \left(\left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11}\right)^{-1} \mathbf{Y}$.

Using theorem 5.11 of the text book, we can say that $\mathbf{Y}^\top \left(\left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11}\right)^{-1} \mathbf{Y}$ follows a non-central chi-squared distribution with non-centrality parameter $\lambda = \mathbf{d}^\top \left(\left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11}\right)^{-1} \mathbf{d}$ and r degrees of freedom, because $\mathbf{C}\left[\mathbf{I}(\boldsymbol{\theta}_{10})^{-1}\right]_{11} = \mathbf{I}_r$ is an idempotent matrix of rank r. This completes the proof that $T_w \sim \chi^2_r(\lambda)$ under the Pitman alternatives.

Problem 6.2

1. The CDF is given by

$$\begin{aligned}
F(x,\sigma) &= \int_0^x f(y,\sigma)dy \\
&= \int_0^x \frac{2y}{\sigma^2}\exp\left(-\frac{y^2}{\sigma^2}\right)dy \\
&= \int_0^{\frac{x^2}{\sigma^2}} \exp(-u)du \qquad \left[\text{change of variable: } u=\frac{y^2}{\sigma^2} \implies du=\frac{2y}{\sigma^2}dy\right] \\
&= 1-\exp\left(-\frac{x^2}{\sigma^2}\right)
\end{aligned}$$

   So, the CDF is given by

$$F(y,\sigma)=\begin{cases}0 & \text{if } y<0\\ 1-\exp\left(-\frac{y^2}{\sigma^2}\right) & \text{if } y>0\end{cases}$$

   Now, $\sigma_1 \neq \sigma_2$ implies $\exp\left(-\frac{y^2}{\sigma_1^2}\right) \neq \exp\left(-\frac{y^2}{\sigma_2^2}\right) \;\; \forall \;\, x$, which implies $F(y,\sigma_1)\neq F(y,\sigma_2)$.

2. From the above expression of the distribution function, it is clear that the CDF of Y does not depend on σ.

3. The log-likelihood based on one observation is given by

$$\ell(y;\sigma) = -2\log\sigma - \frac{y^2}{\sigma^2}$$

   Then, the three derivatives of the log-likelihood is given by

$$\begin{aligned}
\ell'(y;\sigma) &= -\frac{2}{\sigma}+\frac{2y^2}{\sigma^3}\\
\ell''(y;\sigma) &= \frac{2}{\sigma^2}-\frac{6y^2}{\sigma^4}\\
\ell'''(y;\sigma) &= -\frac{4}{\sigma^3}+\frac{24y^2}{\sigma^5}
\end{aligned}$$

   We can see that $l(y;\sigma)$ is infinitely differentiable within the support of Y.

4. To bound the third derivative observe that

$$|\ell'''(y;\sigma)| = \left|-\frac{4}{\sigma^3}+\frac{24y^2}{\sigma^5}\right| \leq \frac{4}{\sigma^3}+\frac{24y^2}{\sigma^5} = g(y)$$

   where, $E(g(y))<\infty$ because

$$\begin{aligned}
E(Y^{2k}) &= \int_0^\infty \frac{2y^{2k+1}}{\sigma^2}\exp\left(-\frac{y^2}{\sigma^2}\right)dy\\
&= \int_0^\infty \sigma^{2k}u^k\exp(-u)du \qquad \left[\text{change of variable: } u=\frac{y^2}{\sigma^2} \implies du=\frac{2y}{\sigma^2}dy\right]\\
&= \sigma^{2k}k! < \infty \quad \forall \;\, k
\end{aligned}$$

5. The expectation of the score function is given by

$$\begin{aligned}
E\left(\ell'(y;\sigma)\right) &= -\frac{2}{\sigma}+\frac{2E(y^2)}{\sigma^3}\\
&= -\frac{2}{\sigma}+\frac{2\sigma^2}{\sigma^3}\\
&= 0
\end{aligned}$$

Fisher information matrix is

$$
\begin{aligned}
E\left(\ell'(y;\sigma)\right)^2 &= E\left[-\frac{2}{\sigma}+\frac{2y^2}{\sigma^3}\right]^2 \\
&= \frac{4}{\sigma^2}-\frac{8}{\sigma^4}E\left(y^2\right)+\frac{4}{\sigma^6}E\left(y^4\right) \\
&= \frac{4}{\sigma^2}-\frac{8}{\sigma^4}\sigma^2+\frac{4}{\sigma^6}\ 2\sigma^4 \\
&= \frac{4}{\sigma^2}
\end{aligned}
$$

On the other hand,

$$
E\left(-\ell''(y;\sigma)\right) = -\frac{2}{\sigma^2}+\frac{6}{\sigma^4}E\left(y^2\right) = \frac{4}{\sigma^2}
$$

This implies that all the five conditions for the asymptotic normality of MLE is satisfied.